Counting metastable states of Ising spin glasses on arbitrary graphs

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(Received 16 January 2008; published 17 April 2008)

Using a field-theoretical representation of the Tanaka-Edwards integral, we develop a method to systematically compute the number N_s of one-spin stable states (local energy minima) of a glassy Ising system with nearest-neighbor interactions and random Gaussian couplings on an arbitrary graph. In particular, we use this method to determine N_s for K-regular random graphs and d-dimensional regular lattices for $d=2,3$. The method works also for other graphs. Excellent accuracy of the results allows us to observe that the number of local energy minima depends mainly on local properties of the graph on which the spin glass is defined.

DOI: 10.1103/PhysRevE.77.041114 PACS number(s): 05.50.+q, 75.10.Nr

I. INTRODUCTION

Glassy systems have nontrivial energy landscapes just like many complex systems observed in nature. The main characteristic of such landscapes is the number N_s of local minima, called also metastable states. Typically, this number grows exponentially with the system size N: $N_s \sim e^{Nf_*}$. The rate of the exponential growth f_* is a fundamental quantity characterizing the complexity of the system. It is, however, very difficult to calculate, and it has been analytically found in only a few cases: for the Ising model with random Gaussian interactions on a complete graph [Sherrington-Kirkpatrick (SK) model] [1,2], on the one-dimensional closed chain [3,4] and for the Ising model with random binary interactions $J=\pm 1$ on K-regular random graphs [5]. Some progress has also been made in estimating the number of local minima and saddle points in the random potential model [6], sharing some features of spin glasses.

In this paper we describe a method to determine f_* for the Ising model with random Gaussian interactions on an arbitrary graph. In particular, we use this method to compute f_* for some K-regular graphs. The idea is to express the number of metastable states N_s in terms of the Tanaka-Edwards integral [1] and then to treat this expression as the partition function of a certain statistical field theory. The logarithm of the partition function can be represented as a sum of connected Feynman diagrams which in turn can be generated and summed on a computer, up to a certain order of the perturbative series. Using some general properties of this series we are able to estimate the value f_* already from the first few orders with a very good accuracy. This allows us to observe that the values of f_* are very similar for regular graphs with different topologies, indicating that the number of metastable states depends mainly on local properties of the graph.

II. DERIVATION OF THE STATISTICAL FIELD THEORY

We consider a system of Ising spins $\sigma_i=\pm 1$, $i=1,\ldots,N$, residing on nodes of a simple graph described by an adjacency matrix $\mathbf{A}$. The graph does not need to be connected. The matrix $\mathbf{A}$ is an $N\times N$ symmetric matrix with $A_{ij}=1$ if i and j are connected by an edge or $A_{ij}=0$ otherwise. The energy of the system is given by

$$E=-\sum_{i>j} A_{ij}J_{ij}\sigma_i\sigma_j, \tag{1}$$

where the coupling constants J_{ij} are random numbers taken from the standardized Gaussian distribution with zero mean and unit variance.

We are interested in counting the number of local minima N_s of the energy (1). A local (metastable) minimum is defined as a configuration of spins $\{\sigma_i\}$ such that a flip of any single spin increases energy. Such a configuration is also called one spin stable. The number of one-spin-stable states is given by [1,5]

$$N_s=\sum_{\sigma_1=\pm 1}\cdots\sum_{\sigma_N=\pm 1}\prod_{i=1}^{N}\theta\left(\sigma_i\sum_j A_{ij}J_{ij}\sigma_j\right), \tag{2}$$

where $\theta(x)$ is a step (Heaviside) function. In Ref. [1] it was shown that the averaging over Gaussian couplings J_{ij} leads to the following concise formula:

$$\langle N_s\rangle=\int_{-\infty}^{\infty}\prod_{k=1}^{N}Dq_k\exp\left(-\frac{1}{2}\sum_{i,j}M_{ij}q_iq_j\right), \tag{3}$$

where

$$Dq_k=\frac{1}{\pi i}\frac{e^{-q_k^2/2}}{q_k-i0^+}dq_k \tag{4}$$

and

$$M_{ij}=\frac{A_{ij}}{\sqrt{k_ik_j}}. \tag{5}$$

Here $k_i=\Sigma_j A_{ij}$ is called the degree of node i. The integral (3) was calculated [1] in the limit of $N\to\infty$ for a complete graph $A_{ij}=1-\delta_{ij}$, yielding the result

$$\ln\langle N_s\rangle\cong Nf_*, \tag{6}$$

with $f_*\approx 0.199\,228$, known also from earlier considerations of the SK model [2].

In this paper we shall propose a systematic method to evaluate this integral also for other graphs. Let us introduce an auxiliary constant g to Eq. (3):

$$\langle N_s\rangle(g) = \int_{-\infty}^{\infty} \prod_{k=1}^{N} Dq_k \exp\left(-\frac{1}{2} g \sum_{i,j} M_{ij} q_i q_j\right). \tag{7}$$

The idea is now to find a systematic way of expanding (7) in powers of g and then to use this series expansion to estimate its value for $g=1$:

$$\langle N_s\rangle = \langle N_s\rangle(g)\big|_{g=1}. \tag{8}$$

Borrowing some techniques from field theory, let us define the following generating function:

$$Z[J] \equiv \int_{-\infty}^{\infty} \prod_{k=1}^{N} Dq_k \exp\left(i \sum_{k} J_k q_k\right), \tag{9}$$

which allows for rewriting Eq. (7) as

$$\langle N_s\rangle(g) = \exp\left(\frac{1}{2} g \sum_{i,j} M_{ij} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j}\right) Z[J]\Bigg|_{J=0}. \tag{10}$$

The function $Z[J]$ has a closed form:

$$Z[J] = \prod_{k=1}^{N} \left[1 + \mathrm{erf}\left(\frac{J_k}{\sqrt{2}}\right)\right] \equiv \prod_{k=1}^{N} \exp\left(\sum_{n=1}^{\infty} c_n J_k^n / n!\right), \tag{11}$$

where $\mathrm{erf}(x)$ is the error function and c_n are cumulants of $1+\mathrm{erf}(J_k/\sqrt{2})$. The coefficients c_n can be easily calculated up to an arbitrary order using a program for symbolic calculations: $c_1=\sqrt{2/\pi}, c_2=-2/\pi, c_3=2(4-\pi)/(\sqrt{2}\pi^{3/2}), \ldots$. Equation (10) can be graphically represented as a sum of vacuum Feynman diagrams of a field theory with the propagator gM_{ij} and Φ_j^n vertices with coupling constants c_n. The Feynman rules to calculate the contribution of a diagram are as follows. To each vertex Φ_j^n, at which n lines meet, we ascribe a factor c_n. The subscript j means that the vertex is decorated by an index $j=1,\ldots,N$ which can be thought of as a color taken from a palette of N possible colors. A line joining two vertices decorated with colors i and j contributes a factor gM_{ij}. Additionally, each diagram has a certain symmetry factor which depends on the shape of the diagram. Finally, one needs to perform the summation over colors.

As usual, the logarithm of Eq. (10),

$$F(g) \equiv \ln\langle N_s\rangle(g), \tag{12}$$

contains only the contribution from connected diagrams. We shall see below that $F(g)$ is an extensive quantity in N. Thus it is convenient to introduce a density $F_\infty(g)$ per spin which for large N becomes a function of g only and can be represented as a power series:

$$F_\infty(g) \equiv \lim_{N\to\infty} \frac{F(g)}{N} = \sum_{l=1}^{\infty} f_l g^l. \tag{13}$$

Our goal is to determine $f_* \equiv F_\infty(1)$, which gives the rate of exponential growth of the number of one-spin-stable configurations for large N.

The coefficients of g^l in the series expansion (13) come from connected Feynman diagrams with l links. In the general case, they must be summed on a computer, because their number grows very fast with l. We need a systematic procedure which allows us to sum diagrams in order to calculate the coefficients f_l. In our approach, such a procedure is as follows: (A) "draw" all possible connected Feynman diagrams with l links; (B) calculate their symmetry factors s; (C) decorate each vertex of the diagram with an index $i=1,\ldots,N$; (D) for each decoration calculate the contribution of the diagram to $F(g)$ as

$$g^l s \prod_{i=1}^{v} c_{n_i} \prod_{\langle ij\rangle} M_{ij}, \tag{14}$$

where v is the number of vertices of the diagram, the first product goes over all vertices and the second one over all links of the diagram; (E) sum contributions of all diagrams and decorations.

Clearly, the procedure described above is not efficient if the original graph is sparse since then many propagators gM_{ij} are zero. Therefore, many of possible N^v decorations give a zero contribution and one wastes time summing many zeros. In particular, all decorations of a diagram having a self-connecting link give no contribution since $M_{ii}=0$. Thus one can omit such diagrams in the sum. One can improve the step (C) of the procedure by concentrating only on decorations that potentially have a chance to contribute. In other words, one should look only for decorations for which propagators M_{ij} do not vanish. This means that i and j must be neighbors on the graph on which the spins reside. Therefore, instead of the step (C) one should take the step (C′) in which one only checks those decorations that are consistent with the graph structure. This can be done iteratively. First, one assigns a label $j=1,\ldots,N$ to one vertex of the diagram. Then one assigns to its neighbors only values i such that $M_{ij}\neq 0$ or equivalently such that j and i are neighbors on the original graph. One repeats this process for neighbors of i and so on, and selects only those labels for which the propagator does not vanish. This speeds up the step (E) of the procedure since now the number of decorations is of order $N\bar{k}^{v-1}$, where $\bar{k}$ is the average node degree of the graph.

The procedure (A-C′-E) works for any graph. For a K-regular graph one can essentially simplify calculations since in this case $M_{ij}=A_{ij}/K$, as it stems from Eq. (5), and thus the elementary contribution (14) to $F(g)$ is

$$\left(\frac{g}{K}\right)^l s \prod_{i}^{v} c_{n_i} \prod_{\langle ij\rangle}^{l} A_{ij}. \tag{15}$$

The last product of A_{ij}'s is either zero or one. Summing over all decorations of a given diagram we get some number P of decorations consistent with the graph structure. Each Feynman diagram has now the following contribution:

$$\left(\frac{g}{K}\right)^l s P \prod_{i}^{v} c_{n_i}. \tag{16}$$

Actually P is the only part of the expression that depends on the graph structure on which spins reside, other quantities can be calculated beforehand. Therefore, we used a C++ program to generate all simple diagrams (without multiple

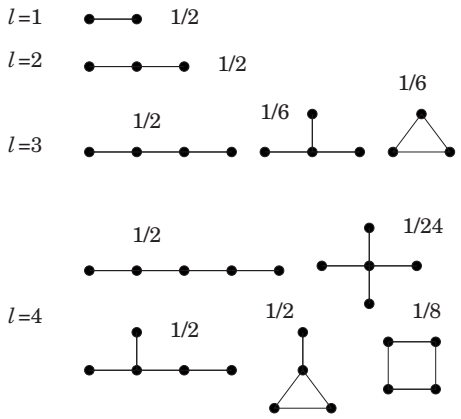

FIG. 1. Feynman diagrams for $l \leq 4$, without multiple links, generated by Eqs. (10) and (11), together with their symmetry factors.

and self-connections) up to $l=13$ and applied the routine NAUTY [7] for isomorphism testing to distinguish different diagrams and to determine their symmetry factors s. In Fig. 1 we show several examples of diagrams for $l \leq 4$. Their number grows very fast with l: for $l=11$ there are 11 461 such diagrams, for $l=12-40\,964$, and for $l=13-153\,786$. The number of all multidiagrams, that is, diagrams with multiple connections, is much larger but luckily there is no need to generate them. Any multidiagram can be obtained from a simple diagram γ by replacing its links by multiple links with a certain multiplicity m. The contribution of all multidiagrams associated with a simple diagram γ is

$$sP \sum_{m_1=1}^{\infty} \cdots \sum_{m_l=1}^{\infty} \left(\frac{g}{K}\right)^{m_1+\cdots+m_l} \prod_{a=1}^{l} \frac{1}{m_a!} \prod_{i=1}^{v} c_{n_i'} \tag{17}$$

where s and P are calculated for γ. The terms in the sum contribute to the order $l'=m_1+\cdots+m_l$ of the expansion of $F(g)$, and l' is the total number of links of the multidiagram. Factors $1/m_a!$ are corrections to the symmetry factor which arise from the fact that one can permute all m_a multiple links joining two vertices without changing the diagram. n_i' is the number of links meeting at vertex i of the multidiagram.

III. A FEW EXAMPLES

Let us illustrate how the method works for spins on a graph with $N=2$ vertices. We have $A_{12}=A_{21}=1$, $A_{11}=A_{22}=0$, and $K=1$. As an example we shall calculate the contribution of the linear diagram with $l=4$ links from Fig. 1. Its total contribution is

$$\frac{1}{2} g^4 c_2^3 c_1^2 P, \tag{18}$$

where the first term accounts for the symmetry factor, the second one is a product of four propagators, the third one is a product of couplings, and the fourth one is the combinatorial factor P depending on the details of the underlying graph's structure encoded in the adjacency matrix $\mathbf{A}$. There are only two possible assignments of labels 1 and 2 to this diagram; see Fig. 2. All other terms vanish and hence $P=2$.

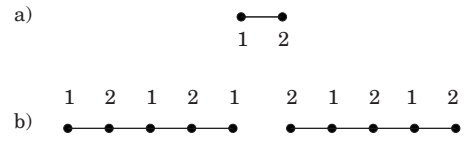

FIG. 2. (a) A graph consisting of only two nodes 1 and 2. (b) The two distinct assignments of the labels 1 and 2 to the linear Feynman diagram for $l=4$. An example of an inconsistent assignment would be, for instance, 11212 since $A_{11}=0$.

This example is somewhat artificial because the number of nodes is small. Now let us suppose that we have a graph where pairs of nodes $2j$ and $2j+1$ are connected, so that the only nonvanishing elements of the adjacency matrix are $A_{2j,2j+1}=A_{2j+1,2j}=1$ where $j=1,\ldots,N/2$. It is a 1-regular graph. One can easily see that the number of decorations P of any Feynman diagram is $P=N$ or $P=0$. Indeed, vertices of a given diagram can be alternately decorated with two consecutive numbers $2j$ and $2j+1$. Thus the choice of a single label specifies automatically the whole decoration. Because this label may assume N values, we have $P=N$. But if the diagram has a loop of odd length, one cannot alternately decorate vertices along this loop, so in this case $P=0$.

Because $P=N$, one can see that also $F(g)$ is proportional to N and hence also $\ln\langle N_s\rangle \sim N$. The proportionality coefficient f_* can be determined in this case analytically by a straightforward calculation of the integral (3):

$$F(g) = \frac{N}{2} \ln\left[\int \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} \mathrm{erfc}^2\left(\sqrt{\frac{g}{2-2g}}x\right)\right], \tag{19}$$

with $\mathrm{erfc}(x)=1-\mathrm{erf}(x)$. We shall use this explicit result to test our method. Using the formula (17) and setting $K=1$, we find that the first coefficient in the expansion of $F(g)$ comes from only one diagram, a line with $l=1$ from Fig. 1, and reads

$$sPc_1^2 = \frac{1}{2} N(\sqrt{2}\pi)^2 = \frac{N}{\pi} = 0.318\,31\ldots \times N. \tag{20}$$

The second coefficient f_2 is a sum of the previous diagram with doubled line, and the one for $l=2$:

$$\frac{1}{4} N\left(\frac{2}{\pi}\right)^2 - \frac{1}{2} N\left(\sqrt{\frac{2}{\pi}}\right)^2 \frac{2}{\pi} = -\frac{N}{\pi^2} = -0.101\,321\ldots \times N. \tag{21}$$

On the other hand, we can calculate f_l numerically from the analytic formula (19):

$$\frac{F(g)}{N} = 0.318\,31g - 0.101\,321g^2 + 0.096\,054g^3 - 0.054\,306g^4 + 0.055\,831g^5 - 0.037\,248g^6 + \cdots. \tag{22}$$

We see that both f_1 and f_2 agree perfectly with those obtained before. Higher coefficients f_l can be calculated by performing the sum (17) on a computer. We checked that all coefficients, up to $l=11$, agree with those from Eq. (19).

In the next section we shall make another cross-check by comparing our results with those for the celebrated SK model. We shall see that the method produces correct values of the coefficients f_l as well as of the limiting value $f_* = F_\infty(1)$.

IV. THE SK MODEL

The SK model [8] is the spin glass (1) on a complete graph, where each node is connected to all other nodes. Before we apply our procedure of diagram summation, let us recall what can be calculated for the SK model using other methods. The integral (7) can be evaluated in the thermodynamic limit [1], leading to

$$F_\infty(g) = \frac{gt^2}{2} - \ln[1 + \mathrm{erf}(gt/\sqrt{2})], \tag{23}$$

where t is a solution to a saddle-point equation

$$t[1 + \mathrm{erf}(gt/\sqrt{2})] = \sqrt{\frac{2}{\pi}} \exp[-(gt/\sqrt{2})^2], \tag{24}$$

with $f_* = F_\infty(1) = 0.199\,228\ldots$. One can find the coefficients f_l by applying Cauchy's differentiation formula and by integrating Eq. (23) numerically. This gives

$$F_\infty(g) = 0.318\,31g - 0.202\,642g^2 + 0.147\,463g^3 - 0.115\,439g^4 + 0.094\,626g^5 - 0.080\,058g^6 + \cdots \tag{25}$$

Let us now calculate the f_l's using the method described in Secs. II and III. The propagator M_{ij} for a complete graph with $N+1$ vertices is $M_{ij} = 1/N$ for any pair of $i \neq j$. It is a K-regular graph with $K=N$, so as one can see from Eq. (16) each link introduces a suppression factor N^{-1}. On the other hand, the combinatorial factor P contains a power N^v, where v is the number of vertices of the diagram. Thus, in the thermodynamic limit, a diagram with l links and v vertices gives a contribution $\sim N^{v-l}$. The exponent $v-l$ is equal to 1 minus the number of closed loops in the diagram. Therefore, in the limit $N \to \infty$ only tree diagrams give nonvanishing contributions. Our task simplifies therefore to summing only tree graphs. Each tree with l links gives the following contribution to f_l:

$$s g^l \prod_{i=1}^{l+1} c_{n_i}, \tag{26}$$

where s is its symmetry factor and n_i is the degree of vertex i.

We performed the summation of all tree diagrams up to $l=11$ on a computer and checked that the values of f_l obtained in this way agree with those obtained from Eq. (23). Again, we see that our method gives correct coefficients f_l.

We should, however, remember that our goal is to find not only the coefficients of the expansion but rather $f_* = F_\infty(1)$ which is a sum of infinitely many coefficients. If we naively terminate the series at some L, $F_L(g) = \sum_{l=1}^{L} f_l g^l$, then, e.g., $F_{11}(1) = 0.220\,701\ldots$ is far away from the true value $0.1992\ldots$, because the series is slowly convergent. Therefore, we need to find a method which allows us to read off the limiting value f_* from the first few coefficients. As we shall see below, one can find a very good estimate of the limiting value $F_\infty(1)$ using some general information about the properties of the series expansion. Let us first observe that the integral (7) and thus also $F_\infty(g)$ is convergent only if the matrix

$$G_{ij} = \delta_{ij} + g M_{ij} \tag{27}$$

is positive definite. In Appendix A we show that it is so for $|g| < 1$ and for $g=1$, and that $\mathbf{G}$ acquires a zero mode for $g=-1$, so the integral (7) is divergent for $g \to -1$. From this we can conclude that the asymptotic behavior of the coefficients f_l has the following form:

$$f_l = \frac{(-1)^l a_l}{l^\alpha}, \tag{28}$$

with all $a_l > 0$, $0 < \alpha \leq 1$, and $\lim_{l\to\infty} a_l \to \mathrm{const}$. From the asymptotic behavior (28) we can deduce the following formula for $F_\infty(1)$:

$$F_\infty(1) \cong F_L(1) + f_L \left(\frac{L}{2}\right)^\alpha [\zeta(\alpha, 1 + L/2) - \zeta(\alpha, 1/2 + L/2)], \tag{29}$$

where $\zeta(\alpha,\beta) = \sum_{k=0}^{\infty} (k+\beta)^{-\alpha}$ is a generalized Riemann zeta function, and α is estimated from the last two coefficients:

$$\alpha \cong -\frac{\left(\ln \frac{-f_L}{f_{L-1}}\right)}{\left(\ln \frac{L}{L-1}\right)}. \tag{30}$$

With the help of formula (29) we can predict now the value of $F_\infty(1) \approx 0.199\,226$ for the SK model. To estimate the maximal error we used a method described in Appendix B. Our final result $f_* = 0.199\,226(5)$ is in excellent agreement with the analytical result cited above.

V. RANDOM K-REGULAR GRAPHS AND CAYLEY TREES

A K-regular graph is a graph with all degrees equal to K. We say that a regular graph is random if its adjacency matrix A is maximally random under the constraints that $A_{ii} = 0$, $A_{ij} = A_{ji}$, and $\sum_j A_{ij} = K$ for all i. If we fix K and let $N \to \infty$, random graphs become sparse and look locally like Cayley trees with degree K, because the average density of finite-length loops goes to zero in this limit. Thus for large N, instead of computing the coefficients f_l by averaging them over many K-regular random graphs one can calculate them for a single Cayley K-tree.

The propagator M_{ij} is simply $1/K$ if nodes i,j are connected, and zero otherwise. Unlike for a complete graph, the contribution from diagrams with loops cannot be neglected. Diagrams with multiple connections also do not vanish. In order to compute the contribution of a simple diagram γ and its multilinked versions we need to evaluate the sum (17). The summation can be performed on a computer with only a

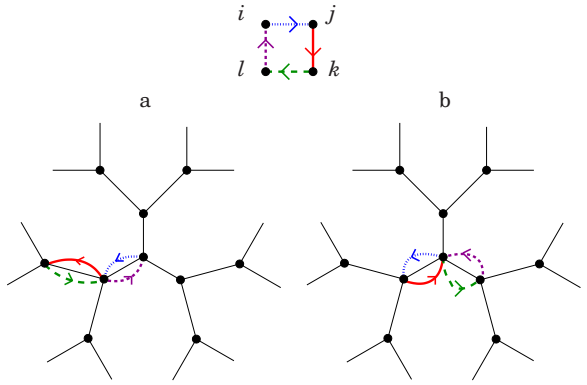

FIG. 3. (Color online) Example of a Feynman diagram (the square on top) superimposed in two different ways on the Cayley tree with $K=3$. Four links are drawn with four different lines and arrows, in order to show how the diagram is put on the graph. When one vertex of the diagram (say the upper left) has fixed index i, there are only six ways of drawing the diagram as in (a) and nine ways as in (b). The picture (a) implies that $j=l$ and there are three ways of choosing j and two for k, while for (b) we have $k=i$ and three possibilities for each index j,l. There is no other way of distributing indices j,k,l, so $P/N=2\times3+3\times3=15$ for the square diagram on the 3-tree.

slight complication as compared to the SK model.

The hardest task in the above is to compute P. Recall that, for a 1-regular graph, P was either N or zero. Now P is simply equal to the number of different ways a given Feynman diagram γ can be superimposed on a K-tree, in such a way that any two neighboring vertices of the diagram are also neighbors on the tree. One expects that the number of such possibilities is of order NK^{v-1}. The factor N comes about since one can set one vertex of the diagram anywhere on the tree. But the second has to be located on one of the K neighbors of the first one. So each time we find a position of a vertex we should check K neighboring nodes on the tree, which gives the factor K^{v-1}. To illustrate this, consider again the linear graph for $l=4$ from Fig. 1. The diagram has $v=5$ vertices. The first vertex can be put anywhere on the tree, but the next one only on one of K neighbors of the first vertex; then the same for the next one, for which we have again K possibilities, etc.; so altogether we have $P=NK^4$. Let us consider now a more complicated example of a square graph from Fig. 1 and $K=3$. Again we have a trivial factor N for choosing the position of the first vertex on the 3-tree. Once it is chosen, we can position the remaining vertices only in 15 ways as shown in Fig. 3. So we have $P=15N$ which is less than $P=3^4N$, because of the constraint coming from a closed loop. Clearly, if a Feynman diagram has no loops then $P=NK^{v-1}=NK^l$. For a diagram with loops one has to consider constraints on possible decorations. This can be done by enumerating all possible labelings and accepting only those that agree with the graph's structure. It can conveniently be done by a computer program. One point must be clarified here—since all P's share the same trivial factor N coming from N possibilities of labeling the initial vertex, in the computer program one can just fix one vertex of the diagram to have some arbitrary label, and consider only decorations consistent with this choice. Next, one multiplies the result by N, which then cancels in the definition of $F_\infty(g)$ so that only numbers independent of N remain.

TABLE I. Values of $f_*=F(1)$, calculated for various graphs with average connectivity K, compared to computer simulations. L stands for the number of coefficients f_l used to estimate $F_\infty(1)$. All simulations were made for $N=10,\ldots,24$, so for relatively small systems. The uncertainty of the last digit is given in parentheses ( ). It was estimated as the standard error in case of computer simulations, and as the maximum error in the way presented in Appendix B for semianalytical calculations.

			f_*, the slope of $\ln\langle N_s\rangle$	
Graph	K	L	Calculated	Simulation
2-regular graph	2	13	0.2414(2)	0.242(2)
3-regular graph	3	12	0.22484(4)	0.226(1)
2D ladder	3	12	0.22568(6)	0.226(2)
4-regular graph	4	11	0.21762(2)	0.219(1)
2D lattice	4	11	0.21808(2)	0.219(2)
3D ladder	4	11	0.21799(2)	0.220(4)
6-regular graph	6	10	0.21101(2)	0.211(1)
3D lattice	6	10	0.21125(1)	
SK model	∞	11	0.199226(5)	0.199(1)

Using this method, we calculated the P's and then the coefficients f_l up to the given order L (typically $L=11$) for $K=2$, 3, 4, and 6. The case $K=1$ has been analyzed in Sec. III. The results are summarized in Table I, where we give the values of f_* for $K\geq2$ with estimated errors and compare them with those obtained by numerical simulations based on enumeration of all metastable states as described in Ref. [9]. To save space, the coefficients f_l are not shown and can be found elsewhere [10].

The agreement with simulations is perfect. For the case $K=2$ there is also a beautiful analytic result [4] to compare with, which gives $f_*=\ln 4/\pi\approx0.241\,56\ldots$. As we see from the table, it agrees with our result within the error bars. We observe also that the rate of convergence of the series expansion for $F_\infty(g)$ grows with K. In other words, for smaller K one should go to larger order L, which is, however, limited by the fast growth of the number of diagrams. This effect is slightly compensated by the fact that the complexity of computing the combinatorial factor $P\sim K^L$ is smaller for smaller K.

VI. REGULAR d-DIMENSIONAL LATTICES

In this section we shall discuss how to calculate f_* for d-dimensional regular lattices, namely, for $d=2$ (square lattice) and 3 (cubic lattice). The lattices are K-regular graphs with $K=2d$, but very special ones. The case $d=1$, that is, a closed chain, gives the same result as a random 2-regular graph, because the latter always contains at least one long chain, whose contribution to N_s in the limit $N\to\infty$ dominates over the contribution coming from shorter chains. In the general case, the only difference as compared to random regular graphs is that now, while calculating P, we shall superimpose Feynman diagrams on the d-dimensional lattice. Again

we see that P is proportional to N since the first vertex can be put anywhere, but once it is fixed we can put a neighboring one on a neighboring site of the lattice and repeat it iteratively for all remaining vertices of the diagram. The calculated values of f_* are given in Table I. It is also worth mentioning that an approximate solution from Ref. [11],

$$f_* \approx 0.1992 + 0.0656K^{-1} + O(K^{-2}), \tag{31}$$

agrees well with our result for $K=6$.

VII. LADDERS

As a further example we consider another particular type of K-regular graph: graphs which we shall call ladders. A ladder is a graph obtained by stacking one above another infinitely many copies of a $(d-1)$-dimensional cube so that corresponding vertices of the copies are aligned on a line and the corresponding vertices of consecutive copies are joined by a link. A $d=2$ ladder is just what one usually would call a ladder except that it is infinitely long. A $d=3$ ladder looks like a bookstand with square-shaped shelves. Such ladders are K-regular graphs with $K=d+1$. It is interesting to compare $f_*=F_\infty(1)$ for ladders with those for random graphs and regular d-dimensional lattices. We adopt the usual scheme. The only thing that changes is again P, because now we have to superimpose diagrams on the ladders. The final results are presented in Table I.

VIII. CONCLUSION

We presented a semianalytic method to estimate the rate f_* of exponential growth of the number of metastable states N_s of the Ising spin glass on different kinds of graphs. We checked that the method reproduces results known analytically, which are available for a few particular cases. The method is based on a diagrammatic representation of the quantity $\ln\langle N_s\rangle(g)$ and on the exact enumeration of all Feynman diagrams up to a given order L. The accuracy of the method improves with growing L, but already for L close to 10 it yields very precise estimates whose uncertainty varies in a range of order $10^{-4}-10^{-6}$ depending on K (see Table I). The results show that the exponent f_* is determined mainly by the degree K. This suggests that the number of local minima N_s depends strongly on local properties of the graph and weakly on its global topology. In other words, important information about the complexity of the energy landscape of the corresponding spin glass is encoded in the short-range properties of the graph. It would be interesting to test if this also holds for other complex systems.

The method presented here can be applied to any type of graph. However, the computational complexity of the method and the dependence of the accuracy of f_* on the order L have to be tested case by case.

In this paper we calculated $\langle N_s\rangle$ for Gaussian J's. Comparing the results to those for binomial J's [5] we see that $\langle N_s\rangle$ significantly depends on the distribution of the J's. It would be quite interesting to investigate the dependence on the distribution of J's in a systematic way by calculating $\langle N_s\rangle$ for some other continuous distributions of J's. One can try to do this by applying the Tanaka-Edwards idea to distributions of the type $p(J)=\int da\, f(a)e^{-J^2/a}$. Another very challenging problem is to calculate higher moments $\langle N_s^k\rangle$ and eventually also $\langle \ln N_s\rangle$.

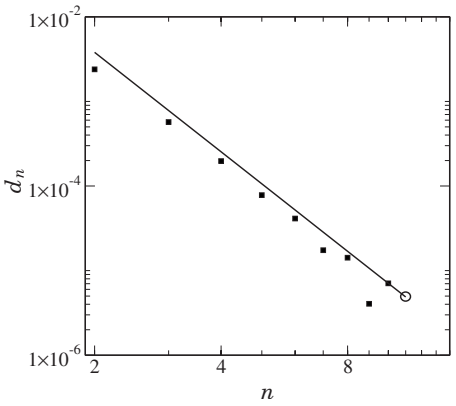

FIG. 4. Illustration of the procedure for estimating errors. We use as an example the data for the SK model, for $L=11$ known coefficients f_l. We plot (n,d_n) (squares) for $n=2,\ldots,10$ in a log-log plot, and then fit a straight line. The line is shifted to be above all the data points. We extrapolate the line to obtain the value for $n=L=11$ (circle) and take this as an uncertainty of $F_\infty(1)$.

ACKNOWLEDGMENTS

We thank W. Janke, A. Krzywicki, and O. Martin for discussion. We thank the EC RTN Network "ENRAGE" for Grant No. MRTN-CT-2004-005616 and the Alexander von Humboldt Foundation for support. Z.B. acknowledges support from the Marie Curie Action Transfer of Knowledge project "COCOS," Grant No. MTKD-CT-2004-517186 and the Polish Ministry of Science and Information Society Technologies Grant No. 1P03B-04029.

APPENDIX A

In this appendix we shall prove that the matrix $\mathbf{G}$ is positive definite for $|g|<1$. We must show that

$$\mathbf{x}^T\mathbf{G}\mathbf{x} = \sum_{i,j} x_i G_{ij} x_j, \tag{A1}$$

is strictly positive for any vector $\mathbf{x}$ with nonzero length. From Eq. (27) we have:

$$\mathbf{x}^T\mathbf{G}\mathbf{x} = \sum_i \left(x_i^2 + g\frac{x_i}{\sqrt{k_i}}\sum_j A_{ij}\frac{x_j}{\sqrt{k_j}} \right). \tag{A2}$$

Introducing variables y_i, $x_i \equiv \sqrt{2k_i}y_i$, we can rewrite the right-hand side as

$$2(1-g)\sum_i k_i y_i^2 + g\sum_{i,j} A_{ij}(y_i+y_j)^2 \tag{A3}$$

$$=2(1+g)\sum_i k_i y_i^2 + g\sum_{i,j} A_{ij}(y_i-y_j)^2. \tag{A4}$$

The formula (A3) shows that the quadratic form (A1) is nonnegative for any $\mathbf{x}$ and for $0<g\leq 1$. It is zero only if $g=1$

and $\mathbf{x}=\mathbf{0}$. Therefore, $F(g)$ tends to a constant for $g\to 1$. On the other hand, Eq. (A4) tells us that for $-1<g<0$ it is positive definite as well. However, for $g=-1$ we see that a zero mode appears. Because of the zero mode the Gaussian integration in Eq. (3) is divergent and consequently $F(g)\to\infty$ for $g\to -1$. These two observations indicate that the radius of convergence of $F(g)$ given by Eq. (13) is 1, and that f_l has the form:

$$f_l = a_l(-1)^l l^{-\alpha}, \tag{A5}$$

where $\alpha\in(0,1]$ and a_l tends to some constant for $l\to\infty$.

APPENDIX B

The estimation of systematic errors like those involved in the calculation of $F_\infty(1)$ from Eq. (29) is not an easy task. We believe that the method described below gives an upper bound on the error of $F_\infty(1)$. Let us slightly adjust the notation for the sake of clarity of the discussion. The value of $F_\infty(1)$ from the left-hand side of Eq. (29) depends on L, for which it has been estimated, so it is convenient to keep a memory of L in the notation. We will denote the value of the estimate by $F_\infty^{(L)}(1)$. Of course it is not the same as $F_L(1)$ which is on the right-hand side of Eq. (29) and which means just a sum of the first L coefficients f_l. The method relies on the following observation. We want to compute the deviation $D_L=F_\infty^{(L)}(1)-F_\infty(1)$ but we do not know the limiting value $F_\infty(1)$. We can, however, compute slightly modified differences $d_n=F_\infty^{(n)}(1)-F_\infty^{(L)}(1)$ for all $n=2,\ldots,L-1$, and say that $D_n\approx d_n$ for L sufficiently large. Now we can plot d_n versus n and extrapolate it to $n=L$ to obtain an estimator of d_L which in turn estimates D_L, giving the error. Because d_n falls with n as a power of n or faster, we can estimate d_L from above by fitting a straight line to the points $(\ln d_n, \ln n)$ and shifting it so that all points lie below it. By extrapolating it to $n=L$ we get the value d_L and use it to estimate the upper bound for the deviation between $F_\infty^{(L)}(1)$ and the true value $F_\infty(1)$. This procedure is illustrated in Fig. 4 for the case of the SK model from Sec. III. How reliable is this method? We checked that the intervals $F(1)\pm\Delta F(1)$ obtained from the $n=2,3,4,\ldots,L-1$ first coefficients f_l always include the value for $n=L$ for all graphs discussed in this paper. We checked also that reducing the error, say, by a factor of 2 would result in many situations for which $F_L(1)$ would lie outside the error bars. This means that the method does not overestimate the error too much.

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